International Journal of Theoretical Physics, Vol. 26, No. 5, 1987

Minimal Supports in Quantum Logics and Hilbert Space

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Received February 21, 1987

It is shown that if a fully atomic, complete orthomodular lattice satisfies a "minimal support condition" (m.s.c.), then it satisfies Piron's axioms, and is thereby shown to be the projection lattice of a generalized Hilbert space. It is shown, conversely, that m.s.c. holds in Hilbert space subspace lattices. The physical justification for m.s.c. is provided in the context of a property lattice $\mathscr{L}(\mathscr{A}, \Sigma)$ for a realistic entity $(\mathscr{A}, \Sigma)$ in the sense of Foulis-Piron-Randall. This context provides a clear focus on key issues in the debate over the appropriateness of requiring quantum logics to be represented over Hilbert spaces.

1. INTRODUCTION

Piron (1976) has provided sufficient conditions for a quantum logic to be the projection lattice of a generalized Hilbert space. Attempts to find physically motivated justification for Piron's axioms, particularly the mathematically motivated "covering axiom," have been at best marginally successful. In this paper we provide a physically justifiable "minimal support condition" that assures that a fully atomic, complete orthomodular lattice will satisfy Piron's axioms. We show, conversely, that the projection lattice of standard Hilbert space satisfies the minimal support condition.

We also address the broader issue of the tradeoff between physical motivation and mathematical convenience in formulations of quantum mechanics as it is brought into sharp focus by Foulis, Piron, and Randall in their operationalist approach (Foulis *et al.*, 1982). They show that imposing the axioms of Hilbert space on their property lattice leads to the result

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0020-7748/87/0500-0435$05.00/0

that their "cannonical map" is a lattice isomorphism from the operational logic to the property lattice, leading to the metaphysically confusing indistinguishability between "propositions" and "properties." Conversely, if the "canonical map" is an isomorphism, and the set of states is not "redundant," then the property lattice is a fully atomic, complete orthomodular lattice. Therefore, if it also satisfies our minimal support conditions, then it satisfies Piron's axioms, leading back to Hilbert space. These results provide a mathematically simple and physically natural framework in which to carry on the debate of the appropriateness of Hilbert space for quantum mechanics.

2. PRELIMINARIES

Suppose $(\mathscr{L}, \leqq, ')$ is a complete, orthomodular lattice (a quantum logic) with zero $0_{\mathscr{L}}$ and one $1_{\mathscr{L}}$. For $p, q \in \mathscr{L}$ we say p is *orthogonal* to q, and we write $p \perp q$, if $p \leqq q'$. We say q *covers* p if $p < q$, and for all $r \in \mathscr{L}$, $p \leqq r \leqq q$ implies $p = r$ or $q = r$. An element $x \in \mathscr{L}$ is called an *atom* if it covers $0_{\mathscr{L}}$. We denote by $A(\mathscr{L})$ the set of all atoms of $\mathscr{L}$. We say $\mathscr{L}$ is *fully atomic* if for every nonzero $p \in \mathscr{L}$, the set $A(p) = \{x \in A(\mathscr{L}) | x \leqq p\}$ is not empty, and $p = \bigvee A(p)$.

Next we briefly review how lattices arise in the operationalist approach to quantum mechanics. We refer the reader to Foulis *et al.* (1982) for further details. Let $\mathscr{A}$ be an orthocoherent manual. We write $X = \bigcup \mathscr{A}$, and refer to X as the set of *outcomes* for $\mathscr{A}$. A subset $S \subseteq X$ is called an *$\mathscr{A}$-support* if (i) $S \cap E \neq \varnothing$ for all $E \in \mathscr{A}$, and (ii) for all $E, F \in \mathscr{A}$, $S \cap (F \backslash E) \neq \varnothing$ iff $S \cap (E \backslash F) \neq \varnothing$. (By $F \backslash E$ we mean $\{x \in F | x \notin E\}$.) A subset satisfying property (i) is called a *transversal.* Property (ii) is called *the exchange condition* for S. As Foulis *et al.* point out, the exchange condition is physically well-motivated for a set S consisting of all outcomes that are possible to occur, if tested, when a physical system characterized by manual $\mathscr{A}$ is in a "realistic state" in the sense of Einstein, Podolsky, and Rosen.

An *entity* $(\mathscr{A}, \Sigma)$ is a manual $\mathscr{A}$ and a set Σ of $\mathscr{A}$-supports such that $\bigcup \Sigma = X$. We call Σ the set of *states* of the entity. A *property* of $(\mathscr{A}, \Sigma)$ is a set $P \subseteq X$ such that $P = \varnothing$ or $P = \bigcup \mathscr{S}$ for some set of states $\mathscr{S} \subseteq \Sigma$. Under the partial ordering of set inclusion the set $\mathscr{L}(\mathscr{A}, \Sigma)$ of all properties forms a complete lattice, *the property lattice*, with zero $0_{\mathscr{L}} = \varnothing$ and one $1_{\mathscr{L}} = X$. A set of states Σ is said to be *not redundant* if $T, S \in \Sigma$ and $T \subseteq S$ implies $T = S$. If Σ is not redundant, then $\mathscr{L}(\mathscr{A}, \Sigma)$ is a fully atomic, complete lattice with Σ as its set of atoms.

For the entity $(\mathscr{A}, \Sigma)$, if $S \in \Sigma$, then event $A \in \mathscr{E}(\mathscr{A})$ is *S-true* if $S \cap E \subseteq A$ whenever $A \subseteq E \in \mathscr{A}$. Given event $A \in \mathscr{E}(\mathscr{A})$, we define $[A] = \bigcup \{S \in \Sigma | A$ is S-true$\}$. Now the operational logic $\Pi(\mathscr{A})$ is an orthomodular poset whose members are equivalence classes of events under the relation: A is equivalent to B iff there is an event C with $A\dot{\bigcup}C$, $B\dot{\bigcup}C \in \mathscr{A}$. If A is equivalent to B, then $[A]=[B]$, so that the *canonical map* $[\ \cdot\ ]$: $A \in \mathscr{E}(\mathscr{A}) \to [A] \in \mathscr{L}(\mathscr{A}, \Sigma)$ can be lifted to become an order-preserving map from $\Pi(\mathscr{A})$ to $\mathscr{L}(\mathscr{A}, \Sigma)$. We shall use the same symbol $[\cdot]$ to refer to the lifted map, relying on the context to avoid confusion for the reader.

Next we review the axioms of C. Piron under which a complete lattice is known to be isomorphic to the lattice of closed subspaces of a (generalized) Hilbert space. In making use of Piron's main result, we shall have available to us an orthocomplementation, which will enable us to state a version of Piron's axioms that are a little less complicated than the version appearing originally in Piron (1976).

Let $\mathscr{L}=(\mathscr{L}, \leqq, ')$ be an orthocomplemented, complete lattice with zero $0_{\mathscr{L}}$ and one $1_{\mathscr{L}}$. We say p, $q \in \mathscr{L}$ are *compatible* if and only if the sublattice in $\mathscr{L}$ generated by $\{p, p', q, q'\}$ is classical (distributive).

Axiom P. If $p, q \in \mathscr{L}$, and $p < q$, then p and q are compatible.

Axiom A1. If $0_{\mathscr{L}} \neq p \in \mathscr{L}$, there exists atom $x \in A(\mathscr{L})$ with $x \leqq p$.

Axiom A2. If $x \in A(\mathscr{L})$ and $q \in \mathscr{L}$ with $x \nleqq q$, then $x \vee q$ covers q.

A lattice satisfying Axiom A1 is said to be *atomic.* It is Axiom A2 that is the central topic of this paper. We shall refer to it as the *covering law.*

The *center* of $\mathscr{L}$ is the set of elements that are compatible with all other elements. We say $\mathscr{L}$ is *irreducible* if and only if the center of $\mathscr{L}$ is $\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$.

The theorem that results from the axioms is:

Theorem 2.1 (Piron). A complete, irreducible, orthocomplemented lattice satisfying Axioms P, A1, and A2, and in which $1_{\mathscr{L}}$ is neither an atom nor the join of two orthogonal atoms, is isomorphic to the lattice of closed subspaces of a (generalized) Hilbert space.

Piron has shown that a general lattice satisfying the above axioms is a direct sum of irreducible lattices (one for each atom in the center), and so it suffices for us to consider irreducible lattices. Physically, a reducible lattice corresponds to one in which there are superselection rules. The assumption about $1_{\mathscr{L}}$ is needed to avoid a pathology in dimension 2.

To complete our preliminaries we state a lemma that will be useful in a later section. Its proof requires a routine application of Zorn's lemma.

Lemma 2.2. Let $(\mathscr{L}, \leqq, ')$ be a complete, atomic lattice with one $1_{\mathscr{L}}$. If $P \subseteq A(\mathscr{L})$ is a pairwise othogonal set of atoms in $\mathscr{L}$, then there exists pairwise orthogonal set E with $P \subseteq E \subseteq A(\mathscr{L})$ and $\bigvee E = 1_{\mathscr{L}}$.

3. THE MINIMAL SUPPORT CONDITION

Throughout this section we suppose that $\mathscr{L}$ is a fully atomic, complete orthomodular lattice with zero $0_{\mathscr{L}}$ and one $1_{\mathscr{L}}$. In the next section we shall see how such lattices arise from rudimentary considerations.

Definitions 3.1.

(i) $\mathscr{A}_{\mathscr{L}} := \{E \subseteq A(\mathscr{L}) | E$ is a pairwise orthogonal set of atoms with $\bigvee E = 1_{\mathscr{L}}\}$.

The members of $\mathscr{A}_{\mathscr{L}}$ are called *operations.*

(ii) $X_{\mathscr{L}} := \bigcup \mathscr{A}_{\mathscr{L}} = A(\mathscr{L})$.

(iii) For $q \in \mathscr{L}$ we define $\mathscr{L}_q := [0, q'] = \{p \in \mathscr{L} | p \leqq q'\}$.

We also define $X_q := A(\mathscr{L}_q)$, the atoms of $\mathscr{L}_q$, and we define $\mathscr{A}_q := \{A \subseteq X_q | A$ is pairwise orthogonal, and $\bigvee A = q'\}$.

It can be shown that $\mathscr{A}_{\mathscr{L}}$ is a coherent manual. Also, for $q \in \mathscr{L}$, $\mathscr{L}_q$ is a fully atomic, complete orthomodular sublattice of $\mathscr{L}$ with zero equal to $0_{\mathscr{L}}$ and with one equal to q'. Thus, $\mathscr{A}_q$ is a manual with outcome set X_q. We shall be concerned with supports on the manuals $\mathscr{A}_{\mathscr{L}}$ and $\mathscr{A}_q$ $(q \in \mathscr{L})$, which we shall call, respectively, $\mathscr{A}_{\mathscr{L}}$-supports and $\mathscr{A}_q$-supports.

For $x \in A(\mathscr{L})$ we write $X_{\mathscr{L}} \backslash x^{\perp}$ for $\{y \in X_{\mathscr{L}} | y \not\perp x\}$. It is not difficult to show that $X_{\mathscr{L}} \backslash x^{\perp}$ is an $\mathscr{A}_{\mathscr{L}}$-support.

Lemma 3.2. If S is an $\mathscr{A}_{\mathscr{L}}$-support and $q \in \mathscr{L}$, and $S \cap X_q \neq \varnothing$, then $S \cap X_q$ is an $\mathscr{A}_q$-support.

Proof. Let $S_q = S \cap X_q$. Suppose E and F are two operations in $\mathscr{A}_q$, so that $\bigvee E = \bigvee F = q'$. We will show that S_q is a transversal and satisfies the exchange condition. Suppose that $E \in \mathscr{A}$, and $S_q \cap E = \varnothing$. Let F be any other operation in $\mathscr{A}_{\mathscr{L}}$ and let G be any set of pairwise orthogonal atoms such that $\bigvee G = q$. The $E \cup G$ and $F \cup G$ are operations in $\mathscr{A}_{\mathscr{L}}$, and since S is a support and $S \cap E = \varnothing$, we have $S \cap F = \varnothing$. Thus, S_q meets no operation in $\mathscr{A}_{\mathscr{L}}$, which contradicts the hypothesis. This proves that S_q is a transversal.

Assume now that there exists $t \in S_q \cap (E \backslash F)$, and we shall show that this implies that $S_q \cap (F \backslash E) \neq \varnothing$. Since $\mathscr{L}$ is fully atomic, there is an orthogonal set A of atoms of $\mathscr{L}$ with $\bigvee A = q$. For every $a \in A$, $e \in E$, we have $e \leqq q' \leqq a'$, so that A is an orthogonal complement of E in $\mathscr{A}_{\mathscr{L}}$. For the same reason, A is also an orthogonal complement of F in $\mathscr{A}_{\mathscr{L}}$. We know that $t \in E$, which implies that $t \neq 0_{\mathscr{L}}$, and $t \leqq q'$. So $t \not\leqq q$, hence $t \notin A$. Thus, $t \in S \cap [(E \dot{\cup} A) \backslash (F \dot{\cup} A)]$. Since S is an $\mathscr{A}_{\mathscr{L}}$-support, we conclude that $S \cap [(F \dot{\cup} A) \backslash (E \dot{\cup} A)] \neq \varnothing$. This implies that $S \cap (F \backslash E) \neq \varnothing$. Since $S \cap F \subseteq S_q$, the proof is complete. $\square$

Definitions 3.3. Let n be a positive integer.

(i) We write rank $(\mathscr{L})=n$ if n is the least integer such that every operation in $\mathscr{A}_{\mathscr{L}}$ consists of n or fewer outcomes.

(ii) We shall say that lattice $\mathscr{L}$ is an *n-lattice* if every operation in $\mathscr{A}_{\mathscr{L}}$ consists of exactly n outcomes.

Note that our rank is one less than the one defined by Piron.

Next we introduce the minimal support condition for $\mathscr{L}$.

Definition 3.4. Lattice $\mathscr{L}$ satisfies the *minimal support condition* (m.s.c.) if and only if for every $x \in A(\mathscr{L})$, and every $q \in \mathscr{L}$, either (a) $\operatorname{rank}(\mathscr{L}_q) \leqq 2$, or (b) if $X_q \backslash x^\perp$ is not empty, then it is minimal (with respect to set inclusion) among all $\mathscr{A}_q$-supports.

Note that if $q=0_{\mathscr{L}}$, then $\mathscr{L}_q=\mathscr{L}$ and $X_q=X_{\mathscr{L}}$, so condition (b) asserts that if $\operatorname{rank}(\mathscr{L}) \geqq 3$, then for every atom $x \in A(\mathscr{L})$, $X_{\mathscr{L}} \backslash x^\perp$ is minimal among all $\mathscr{A}_{\mathscr{L}}$-supports.

The exception for lattices of rank less than three arises because for such a lattice $\mathscr{L}$, no two operations in $\mathscr{A}_{\mathscr{L}}$ intersect, and the exchange condition for supports becomes vacuously satisfied.

The minimal support condition is motivated by two assumptions about the characterization of a physical system by an entity $(\mathscr{A}, \Sigma)$.

First, we assume that the set of states is "unital" in the sense that if $x \in X$ is an experimental outcome, then there should be a state $S \in \Sigma$ in which x is true with certainty. We interpret S as the set of all outcomes that are possible to occur if tested when the system is in state S. Thus, if $y \in S$, then y cannot be orthogonal to x. Hence, $S \subseteq X \backslash x^\perp$. On the other hand, following the principle of Jaynes that S should provide "minimal information," there should not be an outcome in $X \backslash x^\perp$ that is not a member of S. If y were such an outcome, then S would provide not only the information that x is true with certainty, but also the information that y is false with certainty. Therefore, we conclude that $S = X \backslash x^\perp$. In other words, it is quite natural to assume that every $\mathscr{A}$-support of the form $X \backslash x^\perp$ is a state. This is in fact the case in orthodox quantum mechanics in Hilbert space, where even the converse is true: every state is of the form $X \backslash x^\perp$ for some outcome x.

The second consideration leading to m.s.c. is the assumption that nature does not hold back information. If it is assumed that a property lattice used to investigate a system is the correct one for that system, then it must be assumed that the states in that lattice are the ontological states. As such, they should be the minimal properties, the atoms of the property lattice. Therefore, there should be no states that are "hiding" as proper subsets of other states. Nor should any such "hidden" states appear when attention is restricted to a subsystem of the original system by consideration of an order interval of the original lattice. Since all $\mathscr{A}$-supports can be considered properties by virtue of their "testability" provided by the exchange condition, it follows that no state should contain an $\mathscr{A}$-support as a proper subset. The minimal support condition, therefore, is merely an assumption of nonredundancy of ontological states having "indicator outcomes." That is, states of the form $X\backslash x^{\perp}$.

Of course, in practice we deal with lattices that are only imperfect representations of physical systems. The discovery of hidden states in a lattice should serve to confirm the imperfection. As we shall see in Section 5, Hilbert space lattices in orthodox quantum mechanics satisfy m.s.c.

We begin now the main task of this section. We shall show that if $\mathscr{L}$ satisfies m.s.c., then it satisfies the covering law.

Lemma 3.5. If $\mathscr{L}$ satisfies m.s.c. and $z \in \mathscr{L}$, then $\mathscr{L}_z$ satisfies m.s.c.

Proof. Since the lemma is obviously true if $\operatorname{rank}(\mathscr{L}_z) \leqq 2$, we turn immediately to the case $\operatorname{rank}(\mathscr{L}_z) = n \geqq 3$. Suppose $q \in \mathscr{L}_z$, $x \in A(\mathscr{L})$, $\operatorname{rank}((\mathscr{L}_z)_q) \geqq 3$, and $(X_z)_q \backslash x^{\perp} \neq \varnothing$. It is not difficult to verify that $(\mathscr{L}_z)_q = \mathscr{L}_{z \vee q}$, and that $(X_z)_q \backslash x^{\perp} = X_{z \vee q} \backslash x^{\perp}$, which is minimal among $\mathscr{A}_{\mathscr{L}_{z \vee q}}$-supports because $\mathscr{L}$ satisfies m.s.c.$\square$

Before the next lemma we remind the reader than in any orthocomplemented lattice $\mathscr{L}$, if E, $F \in \mathscr{A}_{\mathscr{L}}$, then neither $E \backslash F$ nor $F \backslash E$ can consist of exactly one outcome.

Lemma 3.6. Suppose $n \geqq 3$ is a natural number and that $\mathscr{L}$ is an n-lattice that satisfies m.s.c. If p, $q \in A(\mathscr{L})$ and $p \not\perp q$, then there exists $z \in A(\mathscr{L})$ with $p \vee q \leqq z'$. (That is, $p \perp z$ and $q \perp z$.)

Proof. Let $S = X_{\mathscr{L}} \backslash q^{\perp}$. Then S is a minimal $\mathscr{A}_{\mathscr{L}}$-support, and $p \in S$. Hence $S_1 = S \backslash \{p\}$ either is not a transversal or violates the exchange condition. In the first case, $S \cap E = \{p\}$ for some operation $E \in \mathscr{A}_{\mathscr{L}}$. Thus, $p' \perp q$, which implies that $q = p$, since p is an atom. Any atom z orthogonal to p now serves. In the second case there exist operations E, $F \in \mathscr{A}_{\mathscr{L}}$ with $p \in E \backslash F$ and $S_1 \cap (E \backslash F) = \varnothing$, while $S_1 \cap (F \backslash E) \neq \varnothing$. Since $E \backslash F$ has at least two outcomes, there exists $z \in E \backslash F$ with $z \neq p$. If $z \in S$, then $z \in S_1$, contradicting the fact that $S_1 \cap (E \backslash F) = \varnothing$. Hence, $z \notin S$, so $z \perp q$. Since z, $p \in E$, we also have $z \perp p$, and the proof is complete. $\square$

Theorem 3.7. Let $\mathscr{L}$ satisfy m.s.c. and let n be a natural number. Suppose that $\mathscr{A}_{\mathscr{L}}$ has an operation of cardinality n and no operations of cardinality less than n. Then $\mathscr{L}$ is an n-lattice.

Proof. Our proof is by induction on n.

If $n = 1$, then $1_{\mathscr{L}}$ is an atom and the theorem is trivially true.

Suppose now that $n \geqq 2$ and that the theorem is true for all natural numbers less than n. We also can suppose that $\operatorname{rank}(\mathscr{L}) \geqq 3$, because $n \leqq \operatorname{rank}(\mathscr{L})$, and if $n = \operatorname{rank}(\mathscr{L}) = 2$, then $\mathscr{L}$ is a 2-lattice.

Now there is at least one operation $E \in \mathscr{A}_{\mathscr{L}}$ with $E = \{a_1, \ldots, a_n\} \subseteq A(\mathscr{L})$ and $\bigvee E = 1_{\mathscr{L}}$. We assert that if $F = \{f_1, \ldots, f_m\}$ is also an operation in $\mathscr{A}_{\mathscr{L}}$ with $m > n$, then $F \cap E = \varnothing$. If we suppose to the contrary, we may assume $f_1 = a_1$. Then $\mathscr{L}_{a_1}$ satisfies m.s.c. by Lemma 3.5, and thus, by the induction assumption, $\mathscr{L}_{a_1}$ is an $(n-1)$-lattice. However, we have that

$$\bigvee_{i=2}^{m} f_i = \bigvee_{i=2}^{n} a_i = a_1' = 1_{\mathscr{L}_{a_1}}$$

a contradiction of the fact that all operations in $\mathscr{A}_{\mathscr{L}_{a_1}}$ have the same cardinality.

Let us denote by $\mathscr{A}^n$ the set of operations in $\mathscr{A}_{\mathscr{L}}$ of cardinality n, and let $\mathscr{B} = \mathscr{A}_{\mathscr{L}} \backslash \mathscr{A}^n$. Then $\mathscr{A}_{\mathscr{L}}$ is the disjoint union $\mathscr{A}_{\mathscr{L}} = \mathscr{A}^n \dot{\cup} \mathscr{B}$, and by the preceding paragraph, $X_{\mathscr{L}}$ is also a disjoint union $X_{\mathscr{L}} = X^n \dot{\cup} X_{\mathscr{B}}$, where $X^n = \bigcup \mathscr{A}^n$ and $X_{\mathscr{B}} = \bigcup \mathscr{B}$. Our proof will be complete if we can establish that $X_{\mathscr{B}} = \varnothing$.

Suppose that $x \in X^n$ and $y \in X_{\mathscr{B}}$. By Lemma 2.2, if $z \in X_{\mathscr{L}}$ and $z \perp y$, then $z \in X_{\mathscr{B}}$. Similarly, if $z \in X_{\mathscr{L}}$ and $z \perp x$, then $z \in X^n$. From this it follows that

$$S := X_{\mathscr{L}} \backslash (x^{\perp} \cup y^{\perp}) = (X^n \backslash x^{\perp}) \dot{\cup} (X_{\mathscr{B}} \backslash y^{\perp})$$

From the fact that $\mathscr{A}_{\mathscr{L}} = \mathscr{A}^n \dot{\cup} \mathscr{B}$, it is not difficult to establish that S is an $\mathscr{A}_{\mathscr{L}}$-support. Clearly, $S \subseteq X_{\mathscr{L}} \backslash x^{\perp}$. We shall show that this containment is proper. Since $n > 1$, there exists $z \in X_{\mathscr{B}}$ with $z \perp y$. Then $z \not\perp x$; hence, $z \in X_{\mathscr{L}} \backslash x^{\perp}$, but $z \notin S$. Thus, $X_{\mathscr{L}} \backslash x^{\perp}$ is not a minimal $\mathscr{A}_{\mathscr{L}}$-support, and this contradiction completes our proof. $\square$

Our next lemma is the initial step in an induction argument.

Lemma 3.8. If $\mathscr{L}$ is a 3-lattice that satisfies m.s.c., then $\mathscr{L}$ satisfies the covering axiom.

Proof. Suppose that $x \in A(\mathscr{L})$ and $q \in \mathscr{L}$ with $x \not\leqq q$. We must show that $x \vee q$ covers q. The hypotheses of the lemma imply that every member of $\mathscr{L}$ is either $0_{\mathscr{L}}$, $1_{\mathscr{L}}$, an atom, or a join of two atoms. If $q = r \vee s$ for two atoms r and s, then $x \vee q = x \vee (r \vee s) = 1_{\mathscr{L}}$, which covers q.

Now suppose q is an atom. In case $q \perp x$, there exists $z \in X_{\mathscr{L}}$ such that $\{z, q, x\} \in \mathscr{A}_{\mathscr{L}}$. In case $q \not\perp x$, we know from Lemma 3.6 that there exists $z \in X_{\mathscr{L}}$ orthogonal to both q and x. In either case, then, there exists $z \in X_{\mathscr{L}}$ with $x \vee q \leqq z' < 1_{\mathscr{L}}$. So $x \vee q$ covers q. $\square$

The following theorem completes the induction argument initialized by Lemma 3.8 and establishes the main result of this section.

Theorem 3.9. If n is a natural number greater than 2, and if $\mathscr{L}$ is an n-lattice that satisfies m.s.c., then $\mathscr{L}$ satisfies the covering axiom.

Proof. Lemma 3.8 establishes the result for $n=3$, so we assume $n>3$, and that the theorem is true for all natural numbers less than n. To establish the covering axiom, we suppose $x \in A(\mathscr{L})$, $q \in \mathscr{L}$, and $x \not\leqq q$.

If $x \perp q$ and $r \in \mathscr{L}$ with $q < r \leqq x \vee q$, then there exists $s \in \mathscr{L} \backslash \{0_{\mathscr{L}}\}$ with $s \perp q$ and $s \vee q = r$. Thus,

$$0_{\mathscr{L}} = q' \wedge q \leqq q' \wedge (s \vee q) \leqq q' \wedge (x \vee q)$$

By orthomodularity, we have then that $0_{\mathscr{L}} \leqq s \leqq x$. But we chose s so that $O_{\mathscr{L}} < s$, and we know x is an atom, so $s = x$. Thus, $r = x \vee q$ and so $x \vee q$ covers q.

We suppose now that $x \not\perp q$.

Consider first the case $x \vee q < 1_{\mathscr{L}}$. Then $x \vee q \leqq z'$ for some atom $z \in A(\mathscr{L})$. Thus, $x \leqq z'$ and $q \leqq z'$, so that $x, q \in \mathscr{L}_z$. Now $\mathscr{L}_z$ is an $(n-1)$-lattice and satisfies m.s.c., so by the induction assumption, $x \vee q$ covers q in $\mathscr{L}_z$. Further, if $r \in \mathscr{L}$ and $q < r < x \vee q \leqq z'$, then $r \in \mathscr{L}_z$. This establishes that $x \vee q$ covers q in $\mathscr{L}$.

Next we consider the case $q \vee x = 1_{\mathscr{L}}$. We shall show that q is a co-atom, and hence is covered by $q \vee x$. Suppose $A = \{r_1, \ldots, r_k\} \subseteq A(\mathscr{L})$, A is pairwise orthogonal, and $q = \bigvee A$. If $k = n-1$, then q is a co-atom, so we suppose $k < n-1$ and seek a contradiction.

Now there exist $t_1, \ldots, t_m \in A(\mathscr{L})$ with $m \geqq 2$ such that $\{r_1, \ldots, r_k, t_1, \ldots, t_m\} \in \mathscr{A}_{\mathscr{L}}$. If $t_i \perp x$ for some i, then $q \vee x \leqq t_i' < 1_{\mathscr{L}}$, a contradiction. So we may assume $t_i \not\perp x$ for all i.

Consider first the case $m > 2$. Let $S = X_{\mathscr{L}} \backslash x^{\perp}$, and $S_1 = S \cap X_q = X_q \backslash x^{\perp}$. Then $t_1, t_2 \in S_1$, which ensures that $S_2 := S_1 \backslash \{t_1\} \neq \varnothing$, and by m.s.c., must either not be a transversal or violate the exchange condition. In the first case, $S \cap E = \{t_1\}$ for some $E \in \mathscr{A}_{\mathscr{L}}$. Thus, any atom below t_1' is orthogonal to x. This is a contradiction, since t_2 is below x. In the second case there exist operations $E, F \in \mathscr{A}_q$ with $S_2 \cap (E \backslash F) = \varnothing$, while $S_2 \cap (F \backslash E) \neq \varnothing$. Since $E \backslash F$ has at least two members, there exists $z \in E \backslash F$ with $z \neq t_1$. Then $z \notin S_2$, so $z \in X_q \backslash S_1$; hence $z \notin S$. Thus $z \perp x$. Hence $q \vee x \leqq z' < 1_{\mathscr{L}}$, a contradiction.

Finally, suppose $m = 2$. Since $x \not\perp q$, we know there exists at least one i with $1 \leqq i \leqq k$ and $r_i \not\perp x$. Let $B = A \backslash \{r_i\}$, and let $q_0 = \bigvee B$. Let $S = X \backslash x^{\perp}$, and let $S_1 = S \cap X_{q_0} = X_{q_0} \backslash x^{\perp}$. Since $r_i, t_1, t_2 \in S_1$, we know that $S_2 := S_1 \backslash \{r_i\}$ is a nonempty, proper subset of S_1. If S_2 is not a transversal, then $S_1 \cap E = \{r_1\}$ for some $E \in \mathscr{A}_{\mathscr{L}}$. As above, we conclude that for some i, $t_i \perp x$, a contradiction. Now $\operatorname{rank}(\mathscr{L}_{q_0}) = 3$, so S_2 violates the exchange condition. Then there exist operations E, F in $\mathscr{A}_{q_0}$ with $S_2 \cap (E \backslash F) = \varnothing$, while

$S_2 \cap (F \backslash E) \neq \varnothing$. Since $\mathscr{L}_{q_0}$ is a 3-lattice, $E \cap F = \{p\}$ for $p \perp q_0$. Thus $B \dot{\cup} E$ and $B \dot{\cup} F$ are operations in $\mathscr{A}_{\mathscr{L}}$, and $(B \dot{\cup} E) \cap (B \dot{\cup} F) = B \cup \{p\}$. This implies that $E \backslash F$ contains two outcomes z_1 and z_2, neither of which belongs to S_2. But since S satisfies the exchange condition, and $S \cap [(F \dot{\cup} B) \backslash (E \dot{\cup} \mathrm{B})] \neq \varnothing$, either z_1 or z_2 belongs to $S \cap X_q = S_1$. So $z_1 = r_i$ or $z_2 = r_i$. Assume without loss in generality that $z_1 = r_i$. Then $z_2 \perp q$, and since $z_2 \in X_{q_0} \backslash S$, we have $z_2 \perp x$. Thus, $q \vee x \leqq z_2'$, and this final contradiction completes the proof. □

4. PROPERTY LATTICES AND QUANTUM LOGICS

As we saw in Section 2, lattices arise quite naturally as property lattices $\mathscr{L} = \mathscr{L}(\mathscr{A}, \Sigma)$ for entities $(\mathscr{A}, \Sigma)$. In this section we turn our attention to the question of what properties we may reasonably expect $\mathscr{L}$ to possess.

In one extreme there are examples of natural physical situations for which the property lattice is not even orthocomplemented, hence certainly not orthomodular. See, for example, Aerts (1981) and Meilnik (1976). In the other extreme is orthodox quantum mechanics in Hilbert space in which Σ is the set of stochastic supports, and the canonical map $[\ \cdot\ ]: \Pi(\mathscr{A}) \to \mathscr{L}(\mathscr{A}, \Sigma)$ is a lattice isomorphism that can carry to the property lattice the structure of the operational logic. In this case $\mathscr{L}$ is a fully atomic, complete orthomodular lattice. Moreover, as we remarked in Section 3, it is in this case that $\Sigma = \{X_{\mathscr{L}} \backslash x^{\perp} | x \in X\} = A(\mathscr{L})$.

Foulis, Piron, and Randall have investigated in considerable detail the consequences of the condition that the canonical map is an isomorphism (Foulis *et al.*, 1982; Randall and Foulis, 1982, 1983). Our results show that their attention to this condition is not misplaced. Not only is the condition present in Hilbert space, but we now have the following converse showing how the condition helps to lead back to Hilbert space.

Theorem 4.1. Suppose $(\mathscr{A}, \Sigma)$ is an entity for which Σ is not redundant, and the canonical map is an isomorphism. Suppose also that $\mathscr{A}_{\mathscr{L}}$ contains at least one finite operation, and that $\mathscr{L}$ satisfies m.s.c. Then $\mathscr{L}$ satisfies Piron's axioms; hence, if $\mathscr{L}$ is irreducible and of rank $\geqq 3$, it is isomorphic to the lattice of closed subspaces of a generalized Hilbert space.

Proof. That Axiom P and Axiom A1 are satisfied follows from the orthomodularity of the operational logic, and the fact that $\mathscr{L}$ is fully atomic whenever Σ is not redundant.

Since $\mathscr{A}_{\mathscr{L}}$ has at least one finite operation, there is an integer n and an operation of cardinality n such that no operation has cardinality less than n. Then, by Theorem 3.7, $\mathscr{L}$ is an n-lattice. If $n \leqq 2$, then $\mathscr{L}$ clearly satisfies the covering law, and if $n \geqq 3$, then $\mathscr{L}$ satisfies the covering law by Theorem 3.9. □

This result helps to highlight some key issues in the debate about how restrictive Hilbert space is for quantum mechanics. As we noted in Section 3, if we have any faith at all in the descriptive value of an entity, we should suppose that its set of ontological states is not redundant, and that every $\mathcal{A}_{\mathcal{L}}$-support is a property. So, following Piron and others, if we require the states to be the minimal properties, we should not expect to find $\mathcal{A}_{\mathcal{L}}$-supports as proper subsets of states. The minimal support condition is a natural assumption, therefore, if we have reason to believe that supports of the form $X\backslash x^{\perp}$ $(x \in X)$ are states. That they should be so was argued from a physical point of view in Section 3. But in fact it follows from Theorem 16 in Randall and Foulis (1983), once we known that $[\,\dot{}\,]$ is an isomorphism and Σ is not redundant, that the states are precisely the supports of the form $X\backslash x^{\perp}$. Thus, clearly, m.s.c. is not the most restrictive hypothesis in Theorem 4.1.

Our result therefore provides further evidence that the restrictiveness of the so-called "Mackey-Axiom VII," that quantum logics are isomorphic to subspace lattices of Hilbert spaces, is closely connected to the restrictiveness of requiring that the quantum logic be isomorphic to the property lattice. We note that we have arrived at this conclusion without any mention of stochastic states (probability measures) and the functional analysis often associated with them. On the other hand, Rüttimann (1985) uses functional analysis to arrive at a similar conclusion when he argues that in the presence of a spectral theorm, a quantum logic is isomorphic to the face lattice (of properties) determined by the σ-additive probability measures on the logic.

5. MINIMAL SUPPORTS IN HILBERT SPACE

In this section we show that the minimal support condition holds in the projection lattices of Hilbert spaces. We begin by considering real Hilbert spaces, and after Corollary 5.6 we extend our result to complex and quaternionic spaces.

We begin with real Hilbert space $\mathbb{H}$ of dimension three. Let $\mathcal{A}$ be the "frame manual," the collection of all maximal, orthonormal subsets of $\mathbb{H}$. It will be convenient to picture a typical operation $E \in \mathcal{A}$ as a set of three points on the projective plane (a unit sphere with antipodal points identified) located at the heads of three pairwise orthogonal vectors drawn from the center of the sphere. (See Figure 1.) As usual, we denote $\bigcup \mathcal{A}$ by X. For $x, y \in X$ with $x \neq y$, we define the great circle $\widehat{xy}$ as the set $z^{\perp}$, where z is the unique member of X perpendicular to both x and y.

The operational logic $\Pi(\mathcal{A})$ is isomorphic to the fully atomic, orthomodular, complete lattice of all projection operators on $\mathbb{H}$, which is the quantum logic usually associated with $\mathbb{H}$ in orthodox quantum

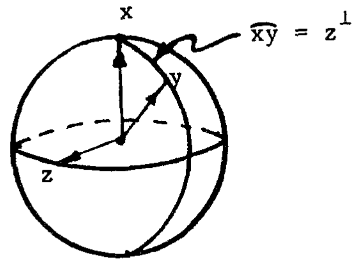

Fig. 1.

mechanics. If we let $\Sigma = \{\text{supp}(\omega) | \omega$ is an $\mathscr{A}$-weight$\}$, then the canonical map $[\,\dot{}\,]$ is a lattice isomorphism, so the property lattice $\mathscr{L} = \mathscr{L}(\mathscr{A}, \Sigma)$ is isomorphic to $\Pi(\mathscr{A})$. Further, Σ is not redundant, so from Theorems 7 and 16 in Randall and Foulis (1982) we have $\Sigma = \{[x] | x \in X\} = \{X \backslash x^{\perp} | x \in X\}$, and Σ is precisely the set of atoms for $\mathscr{L}$. Referring to Figure 1, if $x \in X$ is at the north pole, then $X \backslash x^{\perp}$ is the set of all points in X except the points on the equator. We shall show that $X \backslash x^{\perp}$ is a minimal $\mathscr{A}$-support.

Suppose $x \in X$ and S is an $\mathscr{A}$-support with $S \subseteq X \backslash x^{\perp}$. We can picture x located at the north pole, as in Figure 1. We shall argue that S must be a subset of polar cap. Then we shall show that no such subset can be an $\mathscr{A}$-support.

Lemma 5.1. $x \in S$.

Proof. Suppose $x \notin S$. Select any two points y, z on the equator $x^{\perp}$ with $y \perp z$. Then $y, z \notin X \backslash x^{\perp}$, so $y, z \notin S$. Thus, $E = \{x, y, z\} \in \mathscr{A}$, and $S \cap E = \varnothing$, contradicting the fact that S is an $\mathscr{A}$-support. $\square$

Now let us consider a continuous parametrization of the equator $x^{\perp}$ with real interval $[-a, a)$ $(a > 0)$, so that every point in $x^{\perp}$ corresponds to a real number. To avoid cumbersome notational difficulties, we shall use the same symbol to refer to a number in $[-a, a)$ and to its corresponding point in $x^{\perp}$. For every $y \in X \backslash x^{\perp}$ $(y \neq x)$ we define t_y as the point in $x^{\perp}$ on the great circle $\widehat{xy}$.

Next we define a real-valued function on $[-a, a)$ as follows:

$$f(t) = \inf\{d(t, s) | s \in (S \cap \widehat{xt})\}$$

where $d(t, s)$ is the (shortest) distance along xt from s to t. Our goal is to show that $f'(t)$ exists and equals zero for every $t \in (-a, a)$. This will show that f is a constant function. From there we argue as follows. (See Figure 2.) Suppose $y \in X \backslash x^{\perp}$, and $y \notin S$. Then $y \neq x$ and $f(t_y) \geqq 0$. Let $z \in \widehat{xy}$, so that $f(t_y) = d(z, t_y)$. Then, if $s \in S$, $d(t_s, s) \geqq f(t_s) = f(t_y) = d(z, t_y)$. This will establish that S is a subset of the polar cap determined by z.

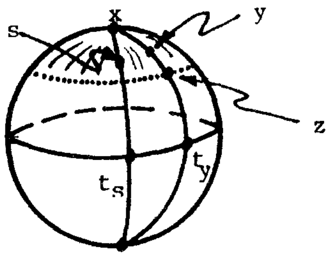

Fig. 2.

We proceed now with the proof that f is a constant function. (We are grateful to James Henle for his contribution suggesting f and the proof that it is a constant function.)

Suppose $y \in X \backslash x^{\perp}$ and $y \neq x$. There is a unique point $u \in x^{\perp}$ with $u \perp y$. We define C_y as the great circle $\widehat{uy}$.

Lemma 5.2. Suppose $y \in X \backslash x^{\perp}$, and $y \notin S$. Then:

(i) $C_y \cap S = \varnothing$.

(ii) If $z \in \widehat{xt_y}$ and $d(t_y, z) \leqq d(t_y, y)$, then $z \notin S$.

Proof. (i) Let u be the unique point in $x^{\perp}$ with $u \perp y$. There is a unique point $v \in X$ such that $E = \{y, u, v\} \in \mathscr{A}$. Now since $u \notin S$ and $y \notin S$, then there cannot exist $w \in S$ with $w \in v^{\perp}$. For if such w existed, one could find a frame $F = \{v, w, r\} \in \mathscr{A}$ so that $S \cap (F \backslash E) \neq \varnothing$, while $S \cap (E \backslash F) = \varnothing$, violating the exchange condition for S. Since $\widehat{uy}$ equals $v^{\perp}$, the proof is complete.

(ii) Suppose $z \in \widehat{xt_y}$ and $d(t_y, z) \leqq d(t_y, y)$. One can construct a geometrical argument to show that there exists $w \in C_y$ with $z \in C_w$. That $z \notin S$ then follows from (i).□

Note that it follows from (ii) that f is not identically zero.

Next we show that $f'(0)$ exists and equals zero. Since our argument does not depend on a particular parametrization for $x^{\perp}$, this will show that f is a constant function.

For each $t \in (-a, a)$ we define z_t as the unique member of $\widehat{xt}$ with $d(t, z_t) = f(t)$. Let us denote by c_t the function defined on $(-a, a)$ by $c_t(r) = d(z_{r,t}, r)$, where $z_{r,t}$ is the only point on $C_{z_t} \cap \widehat{xr}$. Think of c_t as the function whose graph represents the great circle C_{z_t}. Notice that for every t, $c_t(t) = f(t)$.

Lemma 5.3. For every $r \in (-a, a)$, $c_0(r) \leqq f(r)$.

Proof. (Refer to Figure 3.) If $f(0) = 0$, then $z_0 \in x^{\perp}$, and $c_0(r) = 0$ for all r, so the inequality holds. So, we assume $f(0) > 0$. If $w \in \widehat{z_{r,0}, r}$ and $d(w, r) < d(z_{r,0}, r)$, then there exists $y \in \widehat{x0}$ with $d(y, 0) < d(z_0, 0)$ and $w \in C_y$. From

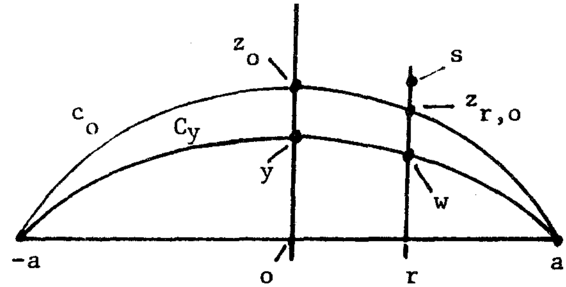

Fig. 3.

the way we define z_0 we know that $y \notin S$. So from Lemma 5.2 we know $w \notin S$. Hence for all $s \in S \cap \widehat{xr}$, $d(s, r) \geqq d(z_{r,0}, r) = c_0(r)$. Thus

$$f(r) = \inf\{d(s, r) | s \in S \cap \widehat{xr}\} \geqq c_0(r) \quad \square$$

Now, $c_0(0) = f(0)$ and c_0 is clearly a differentiable function with a maximum at 0. Thus, from Lemma 5.3 we have

$$\liminf_{r \to 0} \frac{f(r) - f(0)}{r} \geqq \lim_{r \to 0} \frac{c_0(r) - c_0(0)}{r} = c_0'(0) = 0 \tag{1}$$

Lemma 5.4. For every $r \in (-a, a)$, $c_r(0) \leqq f(0)$.

Proof. Suppose $c_r(0) > f(0)$. Let $w \in \widehat{x0}$ with $d(w, 0) = c_r(0)$. Since $f(0) = d(z_0, 0)$, there is a point $s \in S$ on $\widehat{x0}$ with $d(z_0, 0) < d(s, 0) < d(w, 0)$. But in this case there is also a $y \in \widehat{xr}$ with $d(y, r) < d(z_r, r)$ and with $s \in C_y$. Since $y \notin S$, we have by Lemma 5.2 that $s \notin S$, a contradiction. $\square$

It is not difficult to establish that the set $\{c_r' | r \in (-a, a)\}$ is equicontinuous, so that

$$\limsup_{r \to 0} \frac{c_r(r) - c_r(0)}{r} = 0$$

We conclude from Lemma 5.4, then, that

$$\limsup_{r \to 0} \frac{f(r) - f(0)}{r} \leqq \limsup_{r \to 0} \frac{c_r(r) - c_r(0)}{r} = 0 \tag{2}$$

Combining (1) and (2), we have that

$$0 \leqq \liminf_{r \to 0} \frac{f(r) - f(0)}{r} \leqq \limsup_{r \to 0} \frac{f(r) - f(0)}{r} \leqq 0$$

which proves our assertion that $f'(0) = 0$.

Our next goal is to show that no subset of a proper polar cap can be an $\mathscr{A}$-support. This will establish that if S is an $\mathscr{A}$-support and $S \subseteq X \backslash x^{\perp}$, then $S = X \backslash x^{\perp}$.

Suppose $S \subseteq X \backslash x^{\perp}$ and $z \in X \backslash x^{\perp}$ with $d(t_z, z) = f(t_z)$. (Refer to Figure 4.) If we suppose $S \neq X \backslash x^{\perp}$, then it follows from Lemma 5.2(ii) that f is not the zero function, so $f(t_z) > 0$. Let us define the equitorial band $B = \{q \in X | d(t_q, q) < f(t_q)\}$. Notice that $B \cap S = \varnothing$. Since there are points in $S \cap \widehat{xz}$ aribtrarily close to z, there is an $s \in xz \cap S$, and a frame $E = \{u, v, w\}$ with $s \in u^{\perp}$, and $v, w \in B$. But then there is also a frame $F = \{u, s, y\}$, which leads us to a contradiction of the exchange condition, because $S \cap (E \backslash F) = \varnothing$, while $s \in S \cap (F \backslash E)$. This concludes our argument that if $\mathscr{A}$ is the frame manual for Hilbert space of dimension three, then for every $x \in X = \bigcup \mathscr{A}$, $X \backslash x^{\perp}$ is a minimal $\mathscr{A}$-support.

We now consider real Hilbert space $\mathbb{H}$ of arbitrarily dimension greater than two.

Theorem 5.5. Let $\mathbb{H}$ be a Hilbert space of dimension greater than two. Let $\mathscr{F}(\mathbb{H})$ be the frame manual of maximal orthonormal subsets of $\mathbb{H}$, and suppose $x \in X = \bigcup \mathscr{F}(\mathbb{H})$. Then $X \backslash x^{\perp}$ is a minimal $\mathscr{F}(\mathbb{H})$-support.

Proof. Our argument above proves the theorem for $\dim(\mathbb{H}) = 3$, so suppose $\dim(\mathbb{H}) > 3$. Suppose S is an $\mathscr{F}(\mathbb{H})$-support, $S \subseteq X \backslash x^{\perp}$, and $u \in X \backslash x^{\perp}$, and $u \notin S$. Now $u \neq x$, otherwise there exists frame $E = \{x\} \dot{\cup} E_1$ with $E_1 \subseteq x^{\perp}$, contradicting the fact that $S \cap E \neq \varnothing$ for every $E \in \mathscr{F}(\mathbb{H})$. So there exists $y \in X$ with y orthogonal to both u and x. Let $\mathbb{K} = \text{cl.lin.}\{x, u, y\}$.

Now consider submanual $\mathscr{F}(\mathbb{K}^{\perp}) \subseteq \mathscr{F}(\mathbb{H})$, and let G be a frame in $\mathscr{F}(\mathbb{K}^{\perp})$. Suppose $F \in \mathscr{F}(\mathbb{K})$. Then $F \dot{\cup} G \in \mathscr{F}(\mathbb{H})$; thus $S \cap (F \dot{\cup} G) \neq \varnothing$. Further, $G \subseteq x^{\perp}$, so $S \cap G = \varnothing$. It is not difficult to establish from this that $S \cap \mathbb{K}$ is an $\mathscr{F}(\mathbb{K})$-support. But $\mathbb{K}$ is three-dimensional, and because $S \cap \mathbb{K} \subseteq \mathbb{K} \backslash x^{\perp}$ and $u \in (\mathbb{K} \backslash x^{\perp}) \backslash (S \cap \mathbb{K})$, we have contradicted the fact that $\mathbb{K} \backslash x^{\perp}$ must be a minimal $\mathscr{F}(\mathbb{K})$-support. $\square$

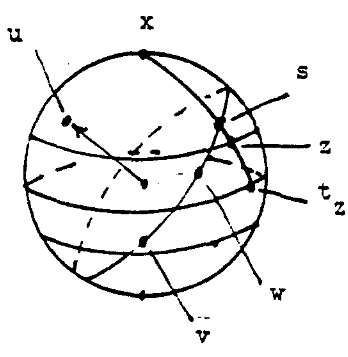

Fig. 4.

Corollary 5.6. Let $\mathbb{H}$ be a Hilbert space of dimension greater than two, and let $\Sigma = \{\text{supp}(\omega) | \omega \text{ is an } \mathcal{F}(\mathcal{H})\text{-weight}\}$. Then the property lattice $\mathcal{L} = \mathcal{L}(\mathcal{F}(\mathbb{H}), \Sigma)$ satisfies the minimal support condition.

Proof. Suppose $x \in A(\mathcal{L})$. From theorem 5.5 we know that $X \backslash x^\perp$ is a minimal $\mathcal{F}(\mathbb{H})$-support. Suppose $q \in \mathcal{L}$. Since $\mathcal{L}$ is principal, we know that $q = [A]$ for some orthonormal set $A \subseteq \mathbb{H}$. If we let $\mathbb{K} = A^\perp$, then $\mathcal{A}_q$ can be identified with $\mathcal{F}(\mathbb{K})$. Thus, either $\text{rank}(\mathcal{A}_q) = \dim(\mathbb{K}) \leqq 2$, or else, by Theorem 5.5., $X_q \backslash x^\perp$ is a minimal $\mathcal{A}_q$-support, if it is not empty. □

Finally, we extend the main result of this section to complex and quaternionic Hilbert spaces.

Let $\mathbb{H}$ be a complex or quaternionic Hilbert space of dimension greater than 2, and let $\mathbb{K}$ be a real Hilbert space with the same dimension as $\mathbb{H}$. Suppose φ: $\mathbb{K} \to \mathcal{H}$ is a real-linear isometry. Then if E is a frame in $\mathcal{F}(\mathbb{K})$, we have that $\varphi(E) := \{\varphi(x) | x \in E\}$ is a frame in $\mathcal{F}(\mathbb{H})$. Further, it is not difficult to establish the following.

Lemma 5.7. If S is an $\mathcal{F}(\mathbb{H})$-support, then $\varphi^{-1}[S]$ is an $\mathcal{F}(\mathbb{K})$-support.

Now suppose $x \in X_{\mathbb{H}}$. We assert that $X_{\mathbb{H}} \backslash x^\perp$ is a minimal $\mathcal{F}(\mathbb{H})$-support. Suppose, to the contrary, that S is an $\mathcal{F}(\mathbb{H})$-support properly contained in $X_{\mathbb{H}} \backslash x^\perp$. Then there exists $y \in X_{\mathbb{H}}$ with $y \not\perp x$ and $y \notin S$. So there exists $z \in x^\perp$ and nonzero scalars α and β such that $y = \alpha x + \beta z$. Further, we can write $\alpha = a\gamma$ and $\beta = b\delta$, where a and b are *real* numbers and γ and δ are unimodular scalars. Now let $x_1 = \gamma x$ and $z_1 = \delta z$, so that $y = ax_1 + bz_1$. Then we can construct an orthonormal basis $B = \{x_1, z_1, h_3, h_4, \dots\}$ for $\mathbb{H}$. Suppose $C = \{k_1, k_2, \dots\}$ is an orthonormal basis for $\mathbb{K}$. Let φ be a real-linear isometry from $\mathbb{K}$ to $\mathbb{H}$ obtained by extending the map from C to B defined by $\varphi(k_1) = x_1$, $\varphi(k_2) = z_1$, and $\varphi(k_i) = h_i$ for all $i \geqq 3$. By Lemma 5.7, $\varphi^{-1}[S]$ is an $\mathcal{F}(\mathbb{K})$-support. Further, it can be established easily that $\varphi^{-1}[S] \subseteq X_{\mathbb{K}} \backslash k_1^\perp$, and that $w := ak_1 + bk_2 \in X_{\mathbb{K}} \backslash k_1^\perp$. Finally, since $\varphi(w) = y \notin S$, we have that $w \notin \varphi^{-1}[S]$, contradicting Theorem 5.5, which asserts that $X_{\mathbb{K}} \backslash k_1^\perp$ must be a minimal $\mathcal{F}(\mathbb{K})$-support.

ACKNOWLEDMENTS

One of the authors (G.S.) acknowledges the financial support of the following agencies of the Brazilian government: the Financiadora de Estudos e Projetos (FINEP) and the Conselho Nacional de Desenvolvimento Cientifico e Tecnologico (CNPq), the latter for a travel grant.

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